What can be a source of growth?

↳ OBSERVATION #1

Decreasing returns + depreciation lead to a steady state.
Growth must come to a stop.

(a) Physical capital (Solow 1956)

$Y = F(K, L)$ with constant returns to (K, L) and decreasing returns to K alone

$$y = \frac{Y}{L} = \frac{F(K,L)}{L} = F\left(\frac{K}{L}, \underbrace{\frac{L}{L}}_{1}\right) := f(k), \quad \text{where } k = \frac{K}{L}$$

- Assume a constant savings rate (Solow model) and a standard capital equation of motion

$$\dot{K} = sY - \delta K$$

$$\frac{\dot{K}}{L} = sy - \delta k$$

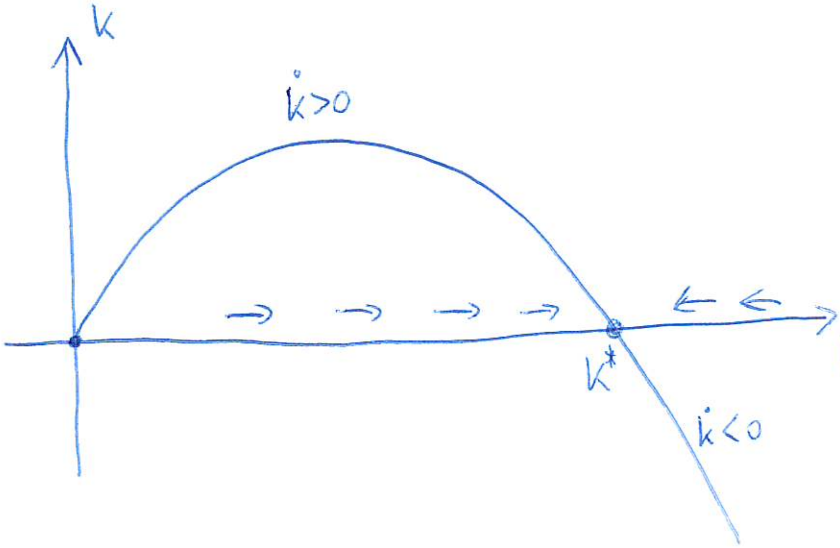
$$\dot{k} = \left(\dot{\frac{K}{L}}\right) = \frac{\dot{K}L - \dot{L}K}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}\cdot\frac{\dot{L}}{L}$$

or $\hat{k} = \hat{K} - \hat{L}$

Assuming a stedy population growth rate
$\hat{L} = n$, $\dot{L} = nL$

$$\dot{k} = \frac{\dot{K}}{L} - kn = sy - (\delta + n)k$$

where $y = f(k)$ — increasing & concave



E.g., Cobb-Douglas production

$$F(K,L) = K^{\alpha} L^{1-\alpha} \Rightarrow f(k) = k^{\alpha}$$

E.g., CES production

$$F(K,L) = \left(\alpha K^{\theta} + (1-\alpha) L^{\theta}\right)^{\frac{1}{\theta}} \Rightarrow f(k) = \left(\alpha k^{\theta} + 1-\alpha\right)^{\frac{1}{\theta}}$$

- Decreasing returns to k & a steady state are guaranteed for the GROSS COMPLEMENTARITY CASE ($\theta<0$) but not the GROSS SUBSTITUTABILITY CASE ($\theta>0$).

Steady state (of the Solow model)

$$\dot{k}=0 \Leftrightarrow sy = (\delta+n)k$$

- E.g., Cobb-Douglas: $sk^{\alpha} = (\delta+n)k \Rightarrow k^{\alpha-1} = \frac{\delta+n}{s} \Rightarrow k^* = \left(\frac{s}{\delta+n}\right)^{\frac{1}{1-\alpha}}$

- E.g., CES ($\theta<0$): $s\left(\alpha k^{\theta} + 1-\alpha\right)^{\frac{1}{\theta}} = (\delta+n)k$

$$s^{\theta}\alpha k^{\theta} + s^{\theta}(1-\alpha) = (\delta+n)^{\theta} k^{\theta}$$

$$\left(\alpha s^{\theta} - (\delta+n)^{\theta}\right) k^{\theta} = -s^{\theta}(1-\alpha)$$

$$k^{\theta} = \frac{s^{\theta}(1-\alpha)}{(\delta+n)^{\theta} - \alpha s^{\theta}}$$

$$k^* = \frac{s(1-\alpha)^{\frac{1}{\theta}}}{\left((\delta+n)^{\theta} - \alpha s^{\theta}\right)^{1/\theta}},$$

under the assumption that $(\delta+n)^{\theta} > \alpha s^{\theta} \Leftrightarrow \delta+n < \alpha^{\frac{1}{\theta}} s$

$$\Leftrightarrow \boxed{s > \frac{\delta+n}{\alpha^{1/\theta}}}.$$

b Human capital (simplified model)

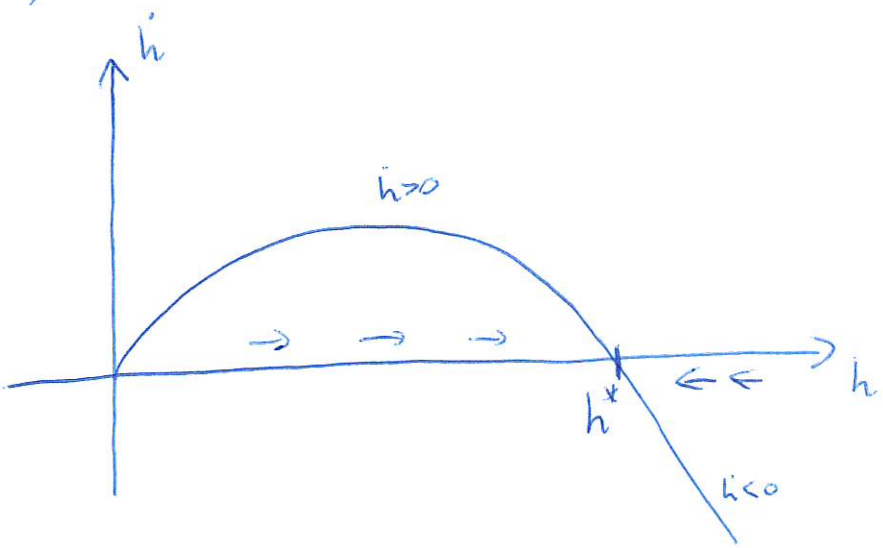
$Y = C = h\ell$, where h – human capital per worker,
ℓ – hours worked per worker, $\ell \in [0,1]$,
$L \equiv 1$ – number of workers.

- Assume à la Solow that $\ell \equiv$ const.
- Let $\varphi(\ell, h)$ be the 'education function' with decreasing returns to h.

$$\boxed{\dot{h} = \varphi(\ell, h) - \delta h}$$

E.g., Cobb-Douglas $\varphi(\ell, h) = (1-\ell)h^{\gamma}$, $\gamma \in (0,1)$
DECREASING RETURNS

$$\dot{h} = (1-\ell)h^{\gamma} - \delta h$$



Steady state

$$\dot{h} = 0 \iff (1-\ell)h^{\gamma-1} = \delta \iff h^* = \left(\frac{1-\ell}{\delta}\right)^{\frac{1}{1-\gamma}}$$

(c) Mankiw-Romer-Weil (1992) model

with <u>both</u> physical & human capital

$$Y = F(K,H,L) \underset{\substack{\uparrow \\ \text{assumed immediately} \\ \text{by MRW}}}{=} K^{\alpha} H^{\beta} L^{1-\alpha-\beta}, \quad \alpha+\beta<1$$

$$\underline{y = k^{\alpha} h^{\beta}}.$$

- Assume <u>identical production functions</u> for physical & human capital as well as the consumption good. And equal <u>depreciation rates</u>.
- Assume constant savings rates à la Solow (s_k, s_h)

$$\begin{cases} \dot{k} = s_k y - (\delta+n)k \\ \dot{h} = s_h y - (\delta+n)h \end{cases}$$

- <u>Steady state:</u>

$$\dot{k} = \dot{h} = 0 \iff \begin{cases} k = \dfrac{s_k y}{\delta+n} \\ h = \dfrac{s_h y}{\delta+n} \end{cases} \Rightarrow \begin{cases} \dfrac{k}{h} = \dfrac{s_k}{s_h} \quad \left(h = \dfrac{k s_h}{s_k}\right) \\ k = \dfrac{s_k k^{\alpha} h^{\beta}}{\delta+n} = \dfrac{s_k k^{\alpha} k^{\beta} \frac{s_h^{\beta}}{s_k^{\beta}}}{\delta+n} \end{cases}$$

$$\Rightarrow \begin{cases} k^{1-\alpha-\beta} = \dfrac{s_h^{\beta} s_k^{1-\beta}}{\delta+n} \\ h = \dfrac{k s_h}{s_k} \end{cases} \Rightarrow \begin{cases} k^* = \left(\dfrac{s_h^{\beta} s_k^{1-\beta}}{\delta+n}\right)^{\frac{1}{1-\alpha-\beta}} \\ h^* = \left(\dfrac{s_k^{\alpha} s_h^{1-\alpha}}{\delta+n}\right)^{\frac{1}{1-\alpha-\beta}} \end{cases}$$

Dynamics around the steady state

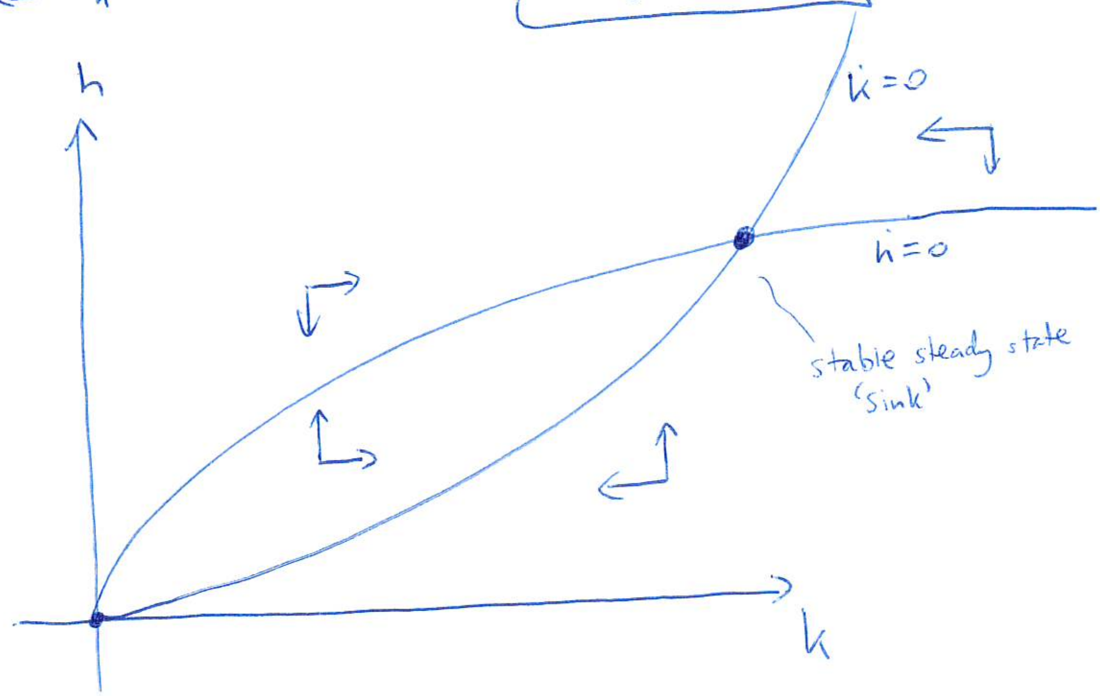
Isoclines

$\dot{k}=0 \iff s_k k^\alpha h^\beta - (\delta+n)k = 0 \iff \underline{h = \left(\frac{(\delta+n)k^{1-\alpha}}{s_k}\right)^{\frac{1}{\beta}}}$

$\dot{h}=0 \iff s_h k^\alpha h^\beta - (\delta+n)h = 0 \iff$

$\iff s_h k^\alpha = (\delta+n)h^{1-\beta} \iff \underline{h = \left(\frac{s_h k^\alpha}{\delta+n}\right)^{\frac{1}{1-\beta}}}$

phase diagram



($\dot{k}=0$ concave if $\frac{1-\alpha}{\beta}<1 \iff 1-\alpha<\beta \iff 1-\alpha-\beta<0$ impossible.)
so $\dot{k}=0$ <u>convex</u>

($\dot{h}=0$ concave if $\frac{\alpha}{1-\beta}<1 \iff \alpha<1-\beta \iff 1-\alpha-\beta>0$ sure.)
so $\dot{h}=0$ <u>concave</u>

OBSERVATION #2

Long-run growth can be imposed EXOGENOUSLY
(hence, 'exogenous growth models')

- (a) Solow model with exogenous growth

$$Y = AF(K,L) \quad \text{or} \quad Y = F(K, AL)$$

$A \approx$ technology level
$\hat{A} = g \approx$ technological progress

E.g., Cobb-Douglas : $Y = K^\alpha \underbrace{(AL)^{1-\alpha}}$ 'Harrod-neutral' tech progress

Balanced growth path ($\hat{Y} \equiv \text{const}$, $\hat{K} \equiv \text{const}$)

$$\dot{K} = sY - \delta K$$

$$\hat{K} = s\frac{Y}{K} - \delta \equiv \text{const} \Rightarrow \frac{Y}{K} \equiv \text{const} \Leftrightarrow \hat{Y} = \hat{K}.$$

Then $\hat{Y} = \hat{K} = \alpha \hat{K} + (1-\alpha)(g+n) \Rightarrow (1-\alpha)\hat{K} = (1-\alpha)(g+n)$

$$\boxed{\hat{Y} = \hat{K} = g+n}$$

$$\left(\hat{k} = \hat{K} - \hat{L} = g+n-n = g.\right)$$

- One may redefine the 'intensive units' as in $k = \frac{K}{AL}$, then $\hat{k} = \hat{K} - g - n = 0$ in the steady state (@BGP)
- Outside of the steady state,

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - g - n = s\frac{Y}{K} - \delta - g - n \Rightarrow \boxed{\dot{k} = sk^\alpha - (\delta+g+n)k}$$

ANALYSIS IS ANALOGOUS.

(b) Human capital

$$Y = C = h\ell$$

$$\mathring{h} = A\varphi(\ell, h) - \delta h,$$

e.g. Cobb-Douglas, $\mathring{h} = A(1-\ell)h^{\gamma} - \delta h$,

with $\hat{A} = g$ — technological progress in the schooling technology

Balanced growth path ($\hat{h} \equiv \text{const}$)

$$\hat{h} = A(1-\ell)h^{\gamma-1} - \delta$$

$$\hat{h} \equiv \text{const} \iff Ah^{\gamma-1} \equiv \text{const} \iff \hat{A} = g = (1-\gamma)\hat{h}$$

$$\iff \boxed{\hat{h} = \frac{g}{1-\gamma}}$$

Then by assumption $\hat{Y} = \hat{C} = \hat{h} = \frac{g}{1-\gamma}$.

- We can rewrite the model in terms of the stationary variable $Ah^{\gamma-1}$, or better $\frac{h^{1-\gamma}}{A}$, or even $\left(\frac{h}{A^{\frac{1}{1-\gamma}}}\right) = \chi$.

$$\hat{\chi} = \hat{h} - \frac{g}{1-\gamma}.$$

$$Ah^{\gamma-1} = \chi^{\frac{1}{\gamma-1}}$$

$$1 + \frac{1}{\gamma-1} = \frac{\gamma}{\gamma-1}$$

$$\mathring{\chi} = \left(\hat{h} - \frac{g}{1-\gamma}\right)\chi = \left((1-\ell)\chi^{\frac{1}{\gamma-1}} - \delta - \frac{g}{1-\gamma}\right)\chi$$

$$\mathring{\chi} = (1-\ell)\chi^{\frac{\gamma}{\gamma-1}} - \left(\delta + \frac{g}{1-\gamma}\right)\chi$$

(c) Mankiw-Romer-Weil model with exogenous growth

- $Y = K^\alpha H^\beta (AL)^{1-\alpha-\beta}$ ← 'Harrod-neutral' tech progress assumed by MRW $\hat{A} = g$

- $\begin{cases} \dot{K} = s_K Y - \delta K \\ \dot{H} = s_h Y - \delta H \end{cases}$

Balanced growth path ($\hat{K} \equiv$ ~~const~~ const, $\hat{H} \equiv$ const, $\hat{Y} \equiv$ const)

$$\begin{cases} \hat{K} = s_K \frac{Y}{K} - \delta \equiv \text{const} \\ \hat{H} = s_h \frac{Y}{H} - \delta \equiv \text{const} \end{cases} \Rightarrow \frac{Y}{K} \equiv \text{const}, \ \frac{Y}{H} \equiv \text{const} \Rightarrow$$

$$\Rightarrow \underline{\hat{Y} = \hat{K} = \hat{H}}.$$

$$\hat{Y} = \alpha \hat{Y} + \beta \hat{Y} + (1-\alpha-\beta)(g+n)$$
$$(1-\alpha-\beta)\hat{Y} = (1-\alpha-\beta)(g+n)$$
$$\underline{\hat{Y} = \hat{K} = \hat{H} = g+n}$$

$\frac{Y}{L}, \frac{K}{L}, \frac{H}{L}$ grow at a rate g.

- One may redefine 'intensive units' as in $y = \frac{Y}{AL}$, $k = \frac{K}{AL}$, $h = \frac{H}{AL}$.

Then
$$\boxed{\begin{aligned} \dot{k} &= s_K y - (n+g+\delta)k \\ \dot{h} &= s_h y - (n+g+\delta)h \end{aligned}}$$

ANALYSIS IS ANALOGOUS.